Pergamon

Computers Math. Applic. Vol. 35, No. 12, pp. 101-108, 1998

Printed in Great Britain
0898-1221/98 $19.00 + 0.00

PII: S0898-1221(98)00100-X

The Noncoercive Quasi-Variational Inequalities Related to Impulse Control Problems

M. BOULBRACHENE*
International Centre for Theoretical Physics
Trieste, Italy

(Received April 1997; accepted June 1997)

Abstract—This paper deals with the numerical analysis of noncoercive quasi-variational inequalities of impulse control problems. Optimally L^∞-error-estimates are derived using qualitative properties of both the continuous and finite element approximation solutions and the notion of subsolutions.

Keywords—Quasi-variational inequality, Impulse control, Subsolution, Finite elements.

1. INTRODUCTION

In this paper, we consider the numerical approximation of the elliptic (stationary) Quasi-Variational Inequality (QVI) arising from stochastic inventory problems with impulse control (see [1]).

This QVI appears in the following formal framework: $\mathcal{A}$ denotes a second-order elliptic differential operator on a bounded smooth domain in $\mathbb{R}^N$, we look for a function u satisfying

$$(\mathcal{P}) \begin{cases} Au - f \leq 0, u - Mu \leq 0, & \text{in } \Omega, \\ (\mathcal{A}u - f)(u - Mu) = 0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

with the addition of suitable boundary conditions.

Naturally, the structure of problem (1.1) is analogous to that of the classical "obstacle problem", where the obstacle function is replaced by an implicit one, depending upon the solution of the problem. The terminology Quasi-Variational-Inequality being chosen is a result of this remark.

In the case studied here, Mu represents a "Cost function" and the prototype of the operators encountered is

$$Mv(x) = k + \inf_{\xi} v(x + \xi); \qquad x \in \Omega, \quad \xi \geq 0; \qquad x + \xi \in \Omega; \qquad k > 0. \tag{1.2}$$

Let $a(\cdot, \cdot)$ be the bilinear form associated with operator $\mathcal{A}$. Then problem $(\mathcal{P})$ formulated in a weak form is as follows.

Find u solution to the following QVI:

$$a(u, v - u) \geq (f, v - u) \qquad v \leq Mu, \quad u \leq Mu, \tag{1.3}$$

$(\ ,\ )$ being the inner product in $L^2(\Omega)$.

*Permanent address: Institut de Mathématiques, Université de Annaba, B.P. 12, Annaba 23000, Algeria.

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But, in modelling of such problems, the coefficient of lowest order of operator $\mathcal{A}$ can be chosen small, for economic motivation, making the bilinear form not coercive. Then there exists $\lambda > 0$ large enough such that (see [1,2])

$$a(v,v) + \lambda \|v\|^2 \geq \gamma, \qquad \|v\|^2, \gamma > 0. \tag{1.4}$$

Set

$$b(u,v) = a(u,v) + \lambda(u,v). \tag{1.5}$$

Then the bilinear form $b(\cdot,\cdot)$ is strongly coercive and therefore, the problem reads as follows.

Find u such that

$$\begin{gathered} b(u, v-u) \geq (f + \lambda u, v-u), \\ v \leq Mu, \qquad u \leq Mu. \end{gathered} \tag{1.6}$$

Problem (1.3) is theoretically well understood, from both analytic and stochastic points of view (see [1]).

The primary aim of our paper is to show that this problem can be properly approximated by a finite element method which turns to be quasi-optimally accurate in L^∞-norm. The convergence orders are carried out by using a subsolution method (see [2,3]), combined with standard piecewise linear finite elements. This method characterizes the continuous solution (respectively, the discrete solution) as the upper bound of the set of continuous subsolutions (respectively, the set of discrete subsolutions).

For the QVI with coercive operator, Cortey-Dumont [4] discussed their numerical approximation. His main tool was the "Bensoussan-Lions algorithm" [1] and the "Hanouzet-Joly estimation" [5].

Our present analysis does not rest on these arguments and, in fact, carries over for several problems. Indeed, the subsolutions method has been used quite successfully in the finite element approximation by the L^∞-norm of noncoercive variational inequalities [2], variational inequalities related to ergodic control problems [3], and Hamilton-Jacobi-Bellman equations [6]. It may also be used for VI with nonlinear source terms [7] and for parabolic variational inequalities as well [8].

The outline of this paper is as follows. We state the continuous Dirichlet problem and study some qualitative properties in Section 2. We consider the discrete problem in Section 3 and set up analogous discrete qualitative properties. In Section 4 we establish auxiliary estimates and give the main results.

2. STATEMENT OF THE CONTINUOUS PROBLEM

2.1. Notations and Assumptions

We are given functions $a_{ij}(x)$, $a_i(x)$, $a_0(x)$, $i,j = 1,\dots,n$ sufficiently smooth, such that

$$\sum a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2, \qquad \forall \xi \in \mathbb{R}^n; \quad \alpha > 0; \quad a_0(x) \geq \beta > 0. \tag{2.1}$$

We define the second-order differential operator

$$\mathcal{A}\varphi = -\sum_{i,j} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial \varphi}{\partial x_j} + \sum_i a_i \frac{\partial \varphi}{\partial x_i} + a_0 \varphi \tag{2.2}$$

and the associated bilinear form: for $u, v \in H_0^1(\Omega)$,

$$a(u,v) = \int_\Omega \left(\sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_i a_i(x) \frac{\partial u}{\partial x_i} v + a_0 uv \right) dx. \tag{2.3}$$

We are given a right hand side f in

$$L^\infty(\Omega); \qquad f \geq f_0 > 0 \tag{2.4}$$

and a nonlinear operator M from $L^\infty(\Omega)$ into itself, defined by

$$M\varphi(x) = k + \inf_\xi \varphi(x+\xi); \qquad x \in \Omega, \quad \xi \geq 0, \quad x+\xi \in \Omega; \quad k > 0. \tag{2.5}$$

This function is called the obstacle of the impulse control. The terminology "impulse control" is justified in [1].

2.2. The Continuous Problem

The following problem is called a "quasi-variational inequality of impulse control".

Find $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} a(u, v-u) \geq (f, v-u) \qquad & v \leq Mu, \quad \forall v \in H_0^1(\Omega),\\ u \leq Mu; \qquad & u \geq 0, \end{aligned} \tag{2.6}$$

or equivalently

$$\begin{aligned} b(u, v-u) \geq (f + \lambda u, v-u) \qquad & v \leq Mu, \quad \forall v \in H_0^1(\Omega),\\ u \leq Mu; \qquad & u \geq 0. \end{aligned} \tag{2.7}$$

2.3. Existence and Uniqueness

Let $u^0 \in H_0^1(\Omega)$ be the unique solution to the equation

$$a(u^0, v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \tag{2.8}$$

and let us define a mapping σ from $L_+^\infty(\Omega)$ into itself ($L_+^\infty(\Omega)$ is the positive cone of $L^\infty(\Omega)$), defined as follows.

For $w \in L_+^\infty(\Omega)$, $\sigma(w)$ is the unique solution to the following coercive variational inequality (VI):

$$\begin{aligned} b(\sigma(w), v-\sigma(w)) \geq (f + \lambda w, v - \sigma(w)) \qquad & v \leq Mw, \quad \forall v \in H_0^1(\Omega),\\ \sigma(w) \leq Mw. & \end{aligned} \tag{2.9}$$

Thanks to [9] the VI (2.9) has one and only one solution.

LEMMA 2.1. *(Cf. [1].) σ is an increasing concave operator which satisfies*

$$\sigma(w) \leq u^0, \qquad \forall w \in L_+^\infty(\Omega) \text{ such that } w \leq u^0. \tag{2.10}$$

In this view, it is natural to consider the following algorithms.

ALGORITHMS. (Cf. [1].)

(1) A decreasing sequence.

Let u^0 be the solution of (2.8), then

$$u^{n+1} = \sigma(u^n), \qquad n = 0, 1, 2, \dots. \tag{2.11}$$

(2) An increasing sequence.

Let $u_0 = 0$, then

$$u_{n+1} = \sigma(u^n), \qquad n = 0, 1, 2, \dots. \tag{2.12}$$

THEOREM 2.1. *(Cf. [1].) Both of the sequences $\{u^n\}_n$ and $\{u_n\}_n$ converge to the unique solution u of QVI (2.6).*

2.4. Regularity of the Solution of QVI (2.6)

THEOREM 2.2. *(Cf. [1].) The solution u of QVI (2.6) satisfies the property:*

$$u \in W^{2,P}(\Omega), \quad 2 \le p < +\infty; \qquad \mathcal{A}u \in L^\infty(\Omega).$$

2.5. A Monotonicity Property

Let $u = \partial(f)$ (respectively, $\tilde{u} = \partial(\tilde{f})$) the solution to QVI (2.6) with right-hand side f (respectively, $\tilde{f}$). Then, we have the following proposition.

PROPOSITION 2.1. *If $f \ge \tilde{f}$, then $\partial(f) \ge \partial(\tilde{f})$.*

PROOF. Let $\tilde{u}^0$ (respectively, $\tilde{u}^0$) the solution of equation (2.8) with right-hand side f (respectively, $\tilde{f}$).

Since $f \ge \tilde{f}$, then by application of standard maximum principle, we get $u^0 \ge \tilde{u}^0$.

Let $u^n = \sigma(u^{n-1})$ and $\tilde{u}^0 = \sigma(\tilde{u}^{n-1})$ and assume that if $f \ge \tilde{f}$, $u^{n-1} \ge \tilde{u}^{n-1}$. Then applying comparisons results in variational inequality, we get: $u^n \ge \tilde{u}^n$. Now, passing to the limit ($n \to +\infty$) we obtain the desired result.

2.6. A Lipschitz Continuous Dependence Property

We keep the precedent notations, i.e., $u = \partial(f)$; $\tilde{u} = \partial(\tilde{f})$.

PROPOSITION 2.2. *Let Proposition 2.1 hold. Then, we have*

$$\|u - \tilde{u}\|_{L^\infty(\Omega)} \le C \left\| f - \tilde{f} \right\|_{L^\infty(\Omega)}.$$

PROOF. Let $\Phi = (1/\beta)\|f - \tilde{f}\|_\infty$; $\|\cdot\|_\infty$ being the L^∞-norm. Then, it is clear that $\tilde{f} \le f + (a_0(x)/\beta)\|f - \tilde{f}\|_\infty$.

So, due to Proposition 2.1, we get

$$\partial\left(\tilde{f}\right) \le \partial(f + a_0 \cdot \Phi) \le \partial(f) + \Phi,$$

which gives

$$\partial\left(\tilde{f}\right) - \partial(f) \le \Phi.$$

Interchanging the roles of f and $\tilde{f}$, we obtain

$$\partial(f) - \partial\left(\tilde{f}\right) \le \Phi,$$

which completes the proof.

2.7. Characterization of the Solution of QVI (2.6) as the Envelope of Subsolutions

DEFINITION 1.1. *w is said to be a subsolution if*

$$\begin{aligned} b(w,v) \le (f + \lambda w, v) \qquad v \ge 0 \quad \forall v \in H_0^1(\Omega), \\ w \le Mw. \end{aligned} \tag{2.13}$$

Let X be the set of such subsolutions. Then, we have the following theorem.

THEOREM 2.3. *(Cf. [1].) The solution of QVI (2.6) is the maximum element of the set X.*

3. STATEMENT OF THE DISCRETE PROBLEM

Let Ω be decomposed into triangles and let τ_h denote the set of those elements; $h > 0$ is the mesh-size.

We assume the triangulation τ_h is regular and quasi-uniform. Let V_h denote the standard piecewise linear finite element space and by $\varphi_i, i = 1, 2, \ldots, m(h)$, the basis functions of the space V_h. Let r_h be the usual restriction operator.

3.1. The Discrete Quasi-Variational Inequality

The discrete QVI consists of solving the following problem: find $u_h \in V_h$ such that

$$\begin{gathered} a(u_h, v - u_h) \geq (f, v - u_h) \qquad v \leq r_h M u_h, \quad \forall v \in V_h, \\ u_h \leq r_h M u_h, \end{gathered} \tag{3.1}$$

or equivalently,

$$\begin{gathered} b(u_h, v - u_h) \geq (f + \lambda u_h, v - u_h) \qquad v \leq r_h M u_h, \quad \forall v \in V_h, \\ u_h \leq r_h M u_h. \end{gathered} \tag{3.2}$$

We can associate with the discrete QVI (3.1) a discrete fixed point mapping σ_h, defined as follows:

$$\begin{gathered} \sigma_h : L^\infty_+(\Omega) \to V_h \\ w \mapsto \sigma_h(w) \end{gathered} \tag{3.3}$$

where $\sigma_h(w)$ is the solution of the following discrete VI:

$$\begin{gathered} b(\sigma_h(w), v - \sigma_h(w)) \geq (f + \lambda w, -\sigma_h(w)) \qquad v \leq r_h M w, \quad \forall v \in V_h, \\ \sigma_h(w) \leq r_h M w. \end{gathered} \tag{3.4}$$

Let $u_h^0 \in V_h$ be the solution of the following equation:

$$a(u_h^0, v) = (f, v), \qquad \forall v \in V_h. \tag{3.5}$$

Then, we have the analog to Lemma 2.1, under the discrete maximum principle assumption (cf. [10]).

LEMMA 3.1. *Let the discrete maximum principle hold, i.e., (angle of triangles of τ_h are $\leq \pi/2$). The the mapping σ_h defined in (3.3),(3.4) is increasing and concave, satisfying: $\sigma_h(w) \leq u_h^0$, $\forall w \in L^\infty_+(\Omega)$ such that $w \leq u_h^0$.*

3.2. Definition of a Discrete Algorithm

Starting from u_h^0 defined in (3.5), (respectively, $u_{0h} = 0$), we define the following:

(1) a discrete decreasing sequence

$$u_h^{n+1} = \sigma_h(u_h^n), \qquad n = 0, 1, 2, \ldots, \tag{3.6}$$

(2) respectively, a discrete increasing sequence

$$u_{n+h} = \sigma_h(u_{nh}), \qquad n = 0, 1, 2, \ldots. \tag{3.7}$$

Then, we obtain as in Theorems 2.1 and 2.3 and Propositions 2.1 and 2.2 their discrete analog, whose proofs are the direct transpositions of the continuous one.

3.3. Existence and Uniqueness of a Discrete Solution

THEOREM 3.1. *Let the discrete maximum hold. Then, both of the sequences* $\{u_h^n\}$ *and* $\{u_{nh}\}$ *converge to the unique solution* u_h *of QVI (3.1).*

3.4. A Monotonicity Property for the Solution of QVI (3.1)

Let $u = \partial_h(f)$; $\tilde{u}_h = \partial_h(\tilde{f})$ the solution of QVI (3.1) with right-hand side f (respectively, $\tilde{f}$). Then, we have the following.

PROPOSITION 3.1. *Under the discrete maximum principle, we have the following: if* $f \geq \tilde{f}$, *then* $\partial_h(f) \geq \partial_h(\tilde{f})$.

3.5. A Lipschitz Continuous Dependence

Keeping the precedent notations, we have the following proposition.

PROPOSITION 3.2.

$$\|u_h - \tilde{u}_h\|_{L^\infty(\Omega)} \leq C. \left\|f - \tilde{f}\right\|_{L^\infty(\Omega)}.$$

3.6. Characterization of the Solution of Discrete QVI (3.1) as the Envelope of Discrete Subsolutions

DEFINITION 3.1. w_h *is said to be a discrete subsolution if*

$$\begin{aligned} b(w_h, \varphi_i) \leq (f + \lambda w_h, \varphi_i), \qquad \forall \varphi_i,\ i = 1, 2, \dots, m(h), \\ w_h \leq r_h M w_h. \end{aligned} \tag{3.8}$$

Let X_h *be the set of discrete subsolutions. Then, we have the following theorem.*

THEOREM 3.2. *Under the discrete maximum principle, the solution of QVI (3.1) is the maximum element of* X_h.

4. FINITE ELEMENT ERROR ANALYSIS

This section is devoted to demonstrate that the proposed method is optimally accurate in $L^\infty(\Omega)$.

Guided by the property results (2.5)–(3.6) of both the continuous and discrete solutions of QVI (2.6) and (3.1), respectively, we first introduce the two following auxiliary problems.

4.1. A Continuous Coercive QVI

Find $\bar{u} \in H_0^1(\Omega)$ such that

$$\begin{aligned} b(\bar{u}^{(h)}, v - \bar{u}^{(h)}) \geq (f + \lambda u_h, v - \bar{u}^{(h)}) \qquad v \leq M\bar{u}, \quad \forall v \in H_0^1(\Omega), \\ \bar{u}_h \leq M\bar{u}^{(h)}, \end{aligned} \tag{4.1}$$

$\bar{u}_h$ being the solution of the discrete QVI (3.1).

4.2. A Discrete Coercive QVI

Find $\bar{u}_h \in V_h$ such that:

$$\begin{aligned} b(\bar{u}_h, v - \bar{u}_h) \geq (f + \lambda u, v - \bar{u}_h) \qquad v \leq r_h M\bar{u}_h, \quad \forall v \in V_h, \\ \bar{u}_h \leq r_h M\bar{u}_h, \end{aligned} \tag{4.2}$$

u being the solution of the continuous QVI (2.6).

4.3. Error Estimate for the Auxiliary Problems

LEMMA 4.1.

(i) $\|\bar{u}_h - u\|_{L^\infty(\Omega)} \leq Ch^2|\log h|^3$,

(ii) $\|\bar{u}^{(h)} - u_h\|_{L^\infty(\Omega)} \leq Ch|\log h|^3$,

where C is a constant independent of h.

PROOF. It is an adaptation of [11].

4.4. Error Estimates for the QVI (2.6)

THEOREM 4.1. *Let Lemma 4.1 hold. Then, we have:*

(i) $\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^2|\log h|^3$,

(ii) $\|u - u_h\|_{W^{1,\infty}(\Omega)} \leq Ch^2|\log h|^3$.

PROOF. The proof of this theorem will be carried out in three steps.

STEP 1. We construct a continuous function $\beta^{(h)}$ such that:

(i) $\beta^{(h)} \leq u$,

(ii) $\|\beta^{(h)} - u\|_\infty \leq Ch^2|\log h|^3$.

Indeed, since $\bar{u}^{(h)}$ is the solution of a QVI, it is also a subsolution:

$$b(\bar{u}^{(h)}, v) \leq (f + \lambda u_h, v) \qquad v \geq 0, \quad v \in H_0^1(\Omega),$$
$$\bar{u}^{(h)} \leq M\bar{u}^{(h)}.$$

Then

$$b(\bar{u}^{(h)}, v) \leq \left(f + \lambda\|u_h - \bar{u}^{(h)}\|_\infty + \lambda\bar{u}^{(h)}, v\right) v \leq 0, \qquad v \in H_0^1(\Omega),$$
$$\bar{u}^{(h)} \leq M\bar{u}^{(h)}.$$

Therefore, by Theorem 2.3, $\bar{u}^{(h)} \in X$ with the right-hand side

$$F = f + \lambda\left\|u_h - \bar{u}^{(h)}\right\|_\infty.$$

Set $U = \partial(F)$. Applying Proposition 2.2, we get

$$\|U - u\|_\infty \leq \lambda \cdot \|u_h - \bar{u}^{(h)}\|_\infty$$

(u being the solution of QVI (2.6)), and by Lemma 4.1(ii): $\bar{u}^{(h)} \leq u + Ch^2|\log h|^3$.

Set $\beta^{(h)} = \bar{u}^{(h)} - Ch^2|\log h|^3$. Then, we clearly have

$$\beta^{(h)} \leq u \quad \text{and} \quad \|\beta^{(h)} - u_h\|_\infty \leq Ch^2|\log h|^3,$$

which completes the proof of Step 1.

STEP 2. We are also able to construct a discrete function α_h such that

$$\alpha_h \leq u_h \quad \text{and} \quad \|\alpha_h - u\|_\infty \leq Ch^2|\log h|^3.$$

We proceed as in Step 1. Indeed, since $\bar{u}_h$ is the solution of QVI (4.2), it is also a discrete subsolution:

$$b(\bar{u}_h, \varphi_i) \leq (f + \lambda u, \varphi_i), \qquad \forall\varphi_i,\ i = 1, 2, \ldots, m(h),$$
$$\bar{u}_h \leq r_h M\bar{u}_h,$$

but

$$\begin{aligned} b(\bar{u}_h, \varphi_i) &\leq (f + (\lambda u - \lambda\bar{u}_h) + \lambda\bar{u}_h, \varphi_i), && \forall\varphi_i, \\ &\leq (f + \lambda\|u - \bar{u}_h\|_\infty + \lambda\bar{u}_h, \varphi_i), && \forall\varphi_i. \end{aligned}$$

Therefore, due to Theorem 3.2, $\bar{u}_h \in X_h$ with right-hand side:

$$F = f + \lambda \|u - \bar{u}_h\|_\infty.$$

Set $U_h = \partial_h(F)$. Then applying Proposition 3.2, we get: $\|U_h - u_h\|_\infty \le \lambda\|u - \bar{u}_h\|_\infty$, and following Lemma 4.1(i), we get

$$\bar{u}_h \le u_h + Ch^2|\log h|^3.$$

Set $\alpha_h = \bar{u}_h - Ch^2|\log h|^3$. Then, clearly,

$$\alpha_h \le u_h \quad \text{and} \quad \|\alpha_h - u\|_\infty \le Ch^2|\log h|^3,$$

which completes the proof of Step 2.

STEP 3. Now, applying results of Steps 1 and 2, we derive the error estimates for QVI (2.6) as follows:

$$\begin{aligned} u_h &\le \beta^{(h)} + Ch^2|\log h|^3 \\ &\le u + Ch^2|\log h|^3 \\ &\le \alpha_h + Ch^2|\log h|^3 \\ &\le u_h + Ch^2|\log h|^3. \end{aligned}$$

Therefore,

$$\|u - u_h\|_{L^\infty(\Omega)} \le Ch^2|\log h|^3$$

and by inverse inequality, we get

$$\|u - u_h\|_{W^{1,\infty}(\Omega)} \le Ch|\log h|^3.$$

REFERENCES

1. A. Bensoussan and J.L. Lions, *Impulse Control and Quasi-Variational Inequalities*, Gauthier-Villars, Paris, (1984).
2. P. Cortey-Dumont, Sur les inéquations variationnelles à opérateurs non coercifs, M^2AN **19**, 195–212 (1985).
3. M. Boulbrachene and H. Sissaoui, On the finite element approximation of variational inequalities related to ergodic control problems, *Int. J. Comp. Math. Appl.* **31**, 137–141 (1996).
4. P. Cortey-Dumont, Approximation numerique d'une IQV liée á des problemes de gestion de stock, *RAIRO, Num. Anal.* **4**, 335–346 (1980).
5. J. Hannouzet and P. Joly, Convergence uniforme des itérés definissant la solution d'une inéquation quasi-variationnelle, *C.R. Acad. Sci., Paris* **286** (Serie A) (1978).
6. M. Boulbrachene, L^∞ error estimate of a finite element method for the Hamilton-Jacobi-Bellman equations, ICTP preprint (1994).
7. K.H. Hoffman and J. Zou, Parallel solution of variational inequality problem with nonlinear source terms, *IMA Journal of Numerical Analysis* **16**, 31–45 (1996).
8. M. Boulbrachene, L^∞ error analysis for parabolic variational inequalities and quasi-variational inequalities, (In preparation).
9. A. Bensoussan and J.L. Lions, *Application des Inequations Variationelles en Controle Stochastique*, Dunod, (1978).
10. P.G. Ciarlet and P.A. Raviart, Maximum principle and uniform convergence for the finite element method, *Comp. Math. in Appl. Mech. and Eng.* **2**, 1–20 (1973).
11. P. Cortey-Dumont, On the finite element approximation in the L^∞-norm of variational inequalities with nonlinear operators, *Numer. Math.* **47**, 45–57 (1985).